6.S891 Lecture 17: Spectral Independence from Zero-Freeness

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In this lecture, we return to (multivariate) zero-freeness of the partition function. We show that it implies spectral independence and hence, fast mixing of Glauber dynamics.

1 The Generating Polynomial and its Zero-Freeness

Let μ be a probability distribution over $2^{[n]}$ (which of course can be identified with $\{\pm 1\}^n$). Define its *generating polynomial* as

$$g_\mu(\boldsymbol{z}) \stackrel{\text{def}}{=} \sum_{S \subseteq [n]} \mu(S) \cdot \boldsymbol{z}^S, \tag{1}$$

where we use the shorthand $\boldsymbol{z}^S \stackrel{\text{def}}{=} \prod_{i \in S} z_i$. This is a *multiaffine* polynomial of degree (at most) n. Its logarithm is essentially the cumulant generating function $\mathcal{L}_\mu$ of μ we previously saw, except with a change of variables to obtain a polynomial. One of the central ideas in the study of the *geometry of polynomials* is the following.

Theme 1.1. *It is fruitful to relate the analytic/algebraic properties of g_μ to the probabilistic/combinatorial properties of μ itself.*

For instance, a consequence of Barvinok's algorithm is that whenever g_μ admits a large zero-free region, then we have efficient algorithms for estimating g_μ.[1] In this lecture, we will be interested in deducing correlation bounds given zero-freeness of g_μ. The following beautiful theorem, which we will not prove, is an example of such a result.

Theorem 1.2 ([BBL09]). *Suppose g_μ is* real stable*, i.e. that $g_\mu \neq 0$ whenever $\operatorname{Im} z_i > 0$ for all $i \in [n]$. Then μ is* negatively correlated *in the sense that*

$$\Pr_{S \sim \mu}[j \in S \mid i \in S] \leq \Pr_{S \sim \mu}[j \in S \mid i \notin S], \qquad \forall i \neq j.$$

Real stability of the generating polynomial of μ is sometimes referred to as the *strongly Rayleigh property*. It turns out to be an extremely robust notion of negative correlation, one which has found many applications; see e.g. [Pem12] and references therein. We also previously saw that negatively correlated distributions on (homogeneous) set systems are 1-spectrally independent. The main result of this lecture is to establish a more direct connection between zero-freeness and spectral independence, one which does not require zero-freeness w.r.t. an entire half-plane.

Recall the following notion of multivariate zero-freeness we used previously.

Definition 1 (Stability). *Let $\Gamma_1, \dots, \Gamma_n \subseteq \mathbb{C}$ be subsets of the complex plane. We say a multivariate polynomial $p(z_1, \dots, z_n)$ is $\Gamma_1 \times \dots \times \Gamma_n$-stable if $p(\boldsymbol{z}) \neq 0$ whenever $z_i \in \Gamma_i$ for all $i = 1, \dots, n$. If $\Gamma_1 = \dots = \Gamma_n = \Gamma$ for some $\Gamma \subseteq \mathbb{C}$, then we simply say p is Γ-stable.*

Theorem 1.3 ([CLV21]; building on [Ali+21]). *Suppose there exists a constant $\delta > 0$ such that g_μ is stable w.r.t. the open radius-δ disk $\mathbb{D}(1, \delta)$ around 1. If in addition the marginals of μ are bounded*

[1] This is not entirely true, since Barvinok's algorithm also requires that that zero-free region contains a point at which computing g_μ is easy. For instance, we do not have FPTAS for estimating arbitrary real stable polynomials.

in the sense that there is a constant $0 < \mathscr{B} \leq 1/2$ such that $\Pr_{S\sim\mu}[i \in S], \Pr_{S\sim\mu}[i \notin S] \geq \mathscr{B}$ for all $i \in [n]$, then

$$\sum_{j=1}^{n} |\Psi_\mu(i \to j)| \leq \frac{4}{\mathscr{B}(1-\mathscr{B})\delta^2}, \qquad \forall i \in [n].$$

In particular, μ is $O(1/\mathscr{B}\delta^2)$-spectrally independent.

Remark 1. The point $\mathbf{1}$ for zero-freeness isn't special. One could look at stability w.r.t. an open radius-δ disk around any other point $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}$, in which case we'd get spectral independence for the tilted distribution $\mu_{\boldsymbol{\lambda}}(S) \propto \mu(S) \cdot \boldsymbol{\lambda}^S$.

Theorem 1.4 ([Ali+21])**.** *Suppose there exists a constant $\alpha > 0$ such that g_μ is stable w.r.t. the sector*

$$S_\alpha \stackrel{\text{def}}{=} \left\{re^{i\theta} : |\theta| < \alpha\pi/2, r > 0\right\}. \tag{2}$$

around the nonnegative real axis with aperture $\alpha\pi$. Then

$$\sum_{j=1}^{n} \left| \Pr_{S\sim\mu}[j \in S \mid i \in S] - \Pr_{S\sim\mu}[j \in S \mid i \notin S] \right| \leq \frac{2}{\alpha}, \qquad \forall i \in [n],$$

and μ is $\left(\frac{2}{\alpha} - 1\right)$-spectrally independent. Furthermore, the same inequality holds for all exponential tilts of μ, i.e. distributions of the form $\mu_{\boldsymbol{\lambda}}(S) \propto \mu(S) \cdot \boldsymbol{\lambda}^S$ for some $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}$.

Remark 2. In the special case $\alpha = 1$, S_α becomes the open right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Polynomials which are stable w.r.t. S_1 are often called *Hurwitz stable.*

The rough intuition behind these statements is the following: Since the correlations of μ are given by second-order derivatives of $\log g_\mu$ at $\mathbf{1}$, these correlations are small if $\log g_\mu(\mathbf{1})$ is "smooth" in some sense. This is only the case if $\mathbf{1}$ is far away from the zeros of g_μ. For more results accommodating more general zero-free regions, see [Ali+21; CLV21].

1.1 Applications

Before we prove Theorems 1.3 and 1.4, let us mention a few applications.

Example 1 (Hardcore Model in Tree Uniqueness). We previously mentioned that Peters–Regts [PR19] established stability of the multivariate independence polynomial $Z_G(\boldsymbol{\lambda})$ of a graph of maximum degree Δ in a neighborhood of the interval $[0, \lambda_c(\Delta))$, where $\lambda_c(\Delta)$ is again the uniqueness threshold w.r.t. the infinite Δ-regular tree. In this regime, Barvinok's algorithm furnishes an FPTAS. Combining this zero-freeness result with Theorem 1.3 yields an alternative proof of $O(1)$-spectral independence of the hardcore Gibbs measure in the uniqueness regime, albeit with worse quantitative bounds.

Example 2 (Monomer-Dimer Model). Recall we previously showed that the univariate matching polynomial $\mathcal{M}_G(z) = \sum_{M\subseteq E \text{ matching}} z^{|M|}$ real-rooted. This is the Heilmann–Lieb Theorem [HL72], and more in depth analysis reveals that for every nonnegative vector of edge weights $\boldsymbol{\lambda} \in \mathbb{R}^E_{\geq 0}$, the multivariate (vertex) matching polynomial

$$\mathcal{M}_G(\boldsymbol{z}) \stackrel{\text{def}}{=} \sum_{\substack{M\subseteq E \\ \text{matching}}} \prod_{e\in M} \lambda_e \prod_{v \text{ unmatched}} z_v,$$

is *Hurwitz stable*; see e.g. [BB09]. Combined with Theorem 1.4, this in particular implies that for any graph $G = (V, E)$, the monomer-dimer Gibbs distribution satisfies the correlation bounds

$$\sum_{v\in V} \left| \Pr_M[v \text{ matched} \mid u \text{ matched}] - \Pr_M[v \text{ matched} \mid u \text{ unmatched}] \right| \leq 2. \tag{3}$$

Note that no assumptions on the degree were made. This result was first proved in a paper of Jeff Kahn [Kah00] using a simple but clever inductive argument.

There are also additional applications of this method to determinantal point processes, even subgraphs, edge covers, edge spin systems (i.e. spin systems on line graphs), etc. [Ali+21; CLV21].

2 A Little Complex Analysis

To formalize the above intuition, we will leverage the following standard fact from complex analysis, which captures the "rigidity" of smooth complex functions.

Lemma 2.1 (Schwarz–Pick)**.** *Let* $f : \mathbb{D}(0,1) \to \mathbb{D}(0,1)$ *is a univariate holomorphic function. Then* $|f'(0)| \le 1 - |f(0)|^2 \le 1$.

In light of the Schwarz–Pick lemma, our strategy will be to construct such a univariate holomorphic function f such that f maps $\mathbb{D}(0,1)$ into itself, and $|f'(0)| \approx \sum_{j=1}^n |\Psi_\mu(i \to j)|$, perhaps up to constants depending on $\mathscr{B}, \delta$.

Proof of Theorem 1.3. Fix an arbitrary $i \in [n]$, and define

$$F_i(\boldsymbol{z}) \stackrel{\text{def}}{=} \frac{\partial_{z_i} \log g_\mu(\boldsymbol{z})}{\Pr_{S\sim\mu}[i \in S] \cdot \Pr_{S\sim\mu}[i \notin S]}.$$

Note that $\partial_{z_j} F_i(\mathbf{1}) = \Psi_\mu(i \to j)$ for all $j \in [n]$. Our goal is to construct appropriate maps $\psi : \mathbb{C} \to \mathbb{C}$ and $\boldsymbol{\varphi} : \mathbb{C} \to \mathbb{C}^n$ such that the composition $f(z) = \psi(F_i(\varphi_1(z), \ldots, \varphi_n(z)))$ is holomorphic and satisfies $f(\mathbb{D}(0,1)) \subseteq \mathbb{D}(0,1)$. If we have such $\psi, \boldsymbol{\varphi}$, then by applying the Schwarz–Pick Lemma,

$$\begin{aligned} 1 &\ge |f'(0)| && \text{(Lemma 2.1)} \\ &= |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \langle \nabla F_i(\boldsymbol{\varphi}(0)), \boldsymbol{\varphi}'(0)\rangle && \text{(Chain Rule)} \\ &= |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \sum_{j=1}^n \Psi_\mu(i \to j) \cdot \varphi_j'(0). \end{aligned}$$

A natural and simple choice is to take $\psi, \boldsymbol{\varphi}$ to be *affine* functions.

- To ensure that f is holomorphic, we use $\boldsymbol{\varphi}$ to map $\mathbb{D}(0,1)$ into the region of stability of g_μ, which we assumed is $\mathbb{D}(1,\delta)$. For convenience, let us allow an epsilon of room. Let

$$\varphi_j(z) \stackrel{\text{def}}{=} 1 + \frac{\delta}{2} s_j z,$$

  where $s_j = \operatorname{sign}(\Psi_\mu(i \to j))$. This ensures that $\varphi_j(\mathbb{D}(0,1)) \subseteq \mathbb{D}(1, \delta/2)$ for all $j \in [n]$,

$$\sum_{j=1}^n \Psi_\mu(i \to j) \cdot \varphi_j'(0) = \frac{\delta}{2} \cdot \sum_{j=1}^n |\Psi_\mu(i \to j)|.$$

  In particular, this choice for $\boldsymbol{\varphi}$ alone already implies

$$\sum_{j=1}^n |\Psi_\mu(i \to j)| \le \frac{2}{\delta \cdot |\psi'(F_i(\mathbf{1}))|}. \tag{4}$$

- Now, we design ψ. In order to apply the Schwarz–Pick Lemma, we need ψ to map the image of $\mathbb{D}(0,1)$ under $F_i \circ \boldsymbol{\varphi}$ back into $\mathbb{D}(0,1)$. This is the tricky part, since we must understand the image of F_i. This is also where it is convenient to have an epsilon of room from the definition of $\boldsymbol{\varphi}$, since we only need to consider $F_i(\mathbb{D}(1,\delta/2))$ instead of $F_i(\mathbb{D}(1,\delta))$.

  Claim 2.2. *The image of* $\mathbb{D}(1,\delta/2)$ *under* F_i *is contained in* $\mathbb{D}\left(0, \frac{2}{\mathscr{B}(1-\mathscr{B})\delta}\right)$.

  Once we have this, an obvious choice for ψ is to scale everything down by a factor of $\frac{\mathscr{B}(1-\mathscr{B})\delta}{2}$. In particular, we let $\psi(z) = \frac{\mathscr{B}(1-\mathscr{B})\delta}{2} z$. Plugging this into Eq. (4) immediately implies the theorem. All that remains is to justify Claim 2.2.

□

Proof of Claim 2.2. Since we assumed the marginal bound $\Pr_{S\sim\mu}[i \in S] \ge \mathscr{B}$, the claim is equivalent to showing that the image of $\mathbb{D}(1,\delta/2)$ under $\boldsymbol{z} \mapsto \partial_i \log g_\mu(\boldsymbol{z})$ is contained in $\mathbb{D}(0, 2/\delta)$.

We go by contradiction, making crucial use linearity of g_μ in each of its variables. Fix $z_1, \dots, z_n \in \mathbb{D}(1, \delta/2)$, and write $y = \partial_i \log g_\mu(\boldsymbol{z})$. We wish to show that $|y| < 2/\delta$. Since

$$y = \partial_i \log g_\mu(\boldsymbol{z}) = \frac{\partial_i g_\mu(\boldsymbol{z})}{g_\mu(\boldsymbol{z})},$$

rearranging yields

$$g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) = 0. \tag{5}$$

Suppose for contradiction that $|y| \geq \frac{2}{\delta}$. Then $\left|-\frac{1}{y}\right| \leq \frac{\delta}{2}$. We use this and Eq. (5) to construct a new vector $\boldsymbol{z}' \in \mathbb{D}(1, \delta)^n$ such that $g_\mu(\boldsymbol{z}') = 0$, contradicting $\mathbb{D}(1, \delta)$-stability of g_μ.

Define $\boldsymbol{z}'$ by $z'_j = z_j$ for all $j \neq i$, and $z'_i = z_i - \frac{1}{y}$. Since $|z_i - 1| < \frac{\delta}{2}$ and $\left|-\frac{1}{y}\right| \leq \frac{\delta}{2}$, we have $\boldsymbol{z}' \in \mathbb{D}(1, \delta)^n$. Furthermore, since g_μ is linear in each of its variables,

$$\begin{aligned} g_\mu(\boldsymbol{z}') &= \underbrace{(g_\mu(\boldsymbol{z}) - z_i \cdot \partial_i g_\mu(\boldsymbol{z}))}_{\text{Monomials without } i} + \underbrace{\left(z_i - \frac{1}{y}\right) \cdot \partial_i g_\mu(\boldsymbol{z})}_{\text{Monomials with } i} \\ &= g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) \\ &= 0 && \text{(By Eq. (5))} \end{aligned}$$

□

3 Better Maps for Sectors

If we impose more structure on our zero-free regions, then we can construct much better $\psi, \boldsymbol{\varphi}$ and prove better bounds. We just need to understand where how the map $y \mapsto -\frac{1}{y}$ changes our zero-free region, and how to map between these regions and the unit disk $\mathbb{D}(0, 1)$.

Proof of Theorem 1.4. Since μ and $\mu_{\boldsymbol{\lambda}}$ have the same generating polynomials up to rescaling the variables by nonnegative coefficients, sector stability also holds for $g_{\mu_{\boldsymbol{\lambda}}}$. Hence, without loss of generality, we just prove spectral independence for μ itself.

Fix an arbitrary $i \in [n]$, and define[2]

$$F_i(\boldsymbol{z}) = \log\left(\frac{\partial_i g_\mu(\boldsymbol{z})}{(1 - z_i \partial_i) g_\mu(\boldsymbol{z})}\right).$$

A direct calculation reveals that $\partial_{z_j} F_i(\mathbf{1}) = \Pr_{S \sim \mu}[j \in S \mid i \in S] - \Pr_{S \sim \mu}[j \in S \mid i \notin S]$ for all $j \in [n]$. We construct appropriate maps $\psi : \mathbb{C} \to \mathbb{C}$ and $\boldsymbol{\varphi} : \mathbb{C} \to \mathbb{C}^n$ such that the composition $f(z) = \psi(F_i(\varphi_1(z), \dots, \varphi_n(z)))$ satisfies the assumptions of the Schwarz–Pick Lemma, since then we'd have

$$1 - |f(0)|^2 \geq |f'(0)| = |\psi'(F_i(\boldsymbol{\varphi}(0)))| \cdot \sum_{j \neq i} \Psi_\mu(i \to j) \cdot \varphi'_j(0). \tag{6}$$

- Since g_μ is stable w.r.t. the sector S_α, we use Möbius transformations and exponential maps instead of affine functions. More specifically, take

$$\begin{gathered} \varphi_j(z) \stackrel{\mathsf{def}}{=} g(s_j z)^\alpha = \left(\frac{1 + s_j z}{1 - s_j z}\right)^\alpha \\ \text{where} \qquad g(x) = \frac{1 + x}{1 - x} \qquad \text{and} \qquad s_j = \operatorname{sign}(\Psi_\mu(i \to j)), \qquad \forall j \in [n]. \end{gathered} \tag{7}$$

  The point is that $\varphi_j(\mathbb{D}(0, 1)) \subseteq S_\alpha$ since the inner Möbius function g maps $\mathbb{D}(0, 1)$ to the right half-plane S_1, and then taking the αth power scales down the angle. A quick calculation

[2] If the numerator inside the logarithm were multiplied by z_i, and if $\boldsymbol{z} \in \mathbb{R}^n_{\geq 0}$, then we'd exactly have the marginal ratio of i under the tilted measure $\mu_{\boldsymbol{z}}$.

reveals that $\varphi_j'(z) = 2s_j\alpha \cdot \left(\frac{1+s_jz}{1-s_jz}\right)^{\alpha-1} \cdot \frac{1}{(1-s_jz)^2}$ and so plugging this into Eq. (6) and using $\boldsymbol{\varphi}(0) = \mathbf{1}$ gives

$$\sum_{j \neq i} |\Psi_\mu(i \to j)| \leq \frac{1}{2\alpha} \cdot \frac{1 - \psi'(F_i(\mathbf{1}))^2}{|\psi'(F_i(\mathbf{1}))|}. \tag{8}$$

- Now let us argue about the image of F_i, which will then tell us how to construct ψ.

Claim 3.1. *For every $z_1, \dots, z_n \in S_\alpha$, we have that*

$$\frac{\partial_i g_\mu(\boldsymbol{z})}{(1 - z_i\partial_i)g_\mu(\boldsymbol{z})} \notin -S_\alpha.$$

In particular, the image of S_α under F_i is contained within the strip

$$\left\{z \in \mathbb{C} : |\operatorname{Im} z| < \left(1 - \frac{\alpha}{2}\right)\pi\right\}. \tag{9}$$

Before we prove this claim, let us finish the proof by constructing ψ. Let

$$\psi(z) = g^{-1}\left(\exp\left(\frac{1/2}{1-\alpha/2} \cdot z\right)\right),$$

where $g^{-1}(z) = \frac{z-1}{z+1}$ is the inverse of the Möbius transformation g we used in the definition of $\boldsymbol{\varphi}$ above. The point is that the inner exponential maps the strip in Claim 3.1 to the right half-plane S_1, and then g^{-1} maps this right half-plane to $\mathbb{D}(0,1)$. Another quick calculation reveals that

$$\psi'(z) = \frac{2\exp\left(\frac{1/2}{1-\alpha/2} \cdot z\right)}{\left(1 + \exp\left(\frac{1/2}{1-\alpha/2} \cdot z\right)\right)^2} \cdot \frac{1/2}{1-\alpha/2} = \frac{1/2}{1-\alpha/2} \cdot \frac{1}{2}\left(1 - \psi(z)^2\right).$$

Combined with Eq. (8) yields

$$\sum_{j \neq i} |\Psi_\mu(i \to j)| \leq \frac{2}{\alpha} - 1.$$

Adding back $\Psi_\mu(i \to i) = 1$ to both sides concludes the proof.

□

Proof of Claim 3.1. Since the image of $-S_\alpha$ under the exponential map $z \mapsto \exp(z)$ is the strip in Eq. (9), the first claim indeed implies the second. Let $y = \frac{\partial_i g_\mu(\boldsymbol{z})}{(1-z_i\partial_i)g_\mu(\boldsymbol{z})}$ and suppose for contradiction that $y \in S_{-\alpha}$. Then $-\frac{1}{y} \in S_\alpha$, whence

$$\begin{aligned} 0 &= (1 - z_i\partial_i)g_\mu(\boldsymbol{z}) - \frac{1}{y} \cdot \partial_i g_\mu(\boldsymbol{z}) && \text{(Rearranging)} \\ &= g_\mu\left(-\frac{1}{y}, \boldsymbol{z}_{-i}\right). && (g_\mu \text{ is multiaffine}) \end{aligned}$$

This contradicts S_α-stability of g_μ. □

References

[Ali+21] Yeganeh Alimohammadi, Nima Anari, Kirankumar Shiragur, and Thuy-Duong Vuong. "Fractionally Log-Concave and Sector-Stable Polynomials: Counting Planar Matchings and More". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2021. Virtual, Italy: Association for Computing Machinery, 2021, pp. 433–446 (cit. on pp. 1, 2).

[BB09] Julius Borcea and Petter Brändén. "The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications". In: *Communications on Pure and Applied Mathematics* 62 (12 2009), pp. 1595–1631 (cit. on p. 2).

[BBL09] Julius Borcea, Petter Brändén, and Thomas M. Liggett. "Negative dependence and the geometry of polynomials". In: *Journal of the American Mathematical Society* 22.2 (2009), pp. 521–567 (cit. on p. 1).

[CLV21] Zongchen Chen, Kuikui Liu, and Eric Vigoda. "Spectral Independence via Stability and Applications to Holant-Type Problems". In: *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*. FOCS 2021. 2021, pp. 149–160 (cit. on pp. 1, 2).

[HL72] Ole J. Heilmann and Elliott H. Lieb. "Theory of monomer-dimer systems". In: *Communications in Mathematical Physics* 25 (1972), pp. 190–232 (cit. on p. 2).

[Kah00] Jeff Kahn. "A Normal Law for Matchings". In: *Combinatorica* 20 (2000), pp. 339–391 (cit. on p. 2).

[Pem12] Robin Pemantle. "Hyperbolicity and stable polynomials in combinatorics and probability". In: *arXiv preprint arXiv:1210.3231* (2012) (cit. on p. 1).

[PR19] Han Peters and Guus Regts. "On a conjecture of Sokal concerning roots of the independence polynomial". In: *The Michigan Mathematical Journal* 68.1 (2019), pp. 33–55 (cit. on p. 2).